Math 4210 Tutorial 11.

1. Recall $C_E(t, S_t) = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$ in Course note 6, compute

$$\Delta = \partial_x C_E(t, S_t) = N(d_1).$$

and

$$\Gamma = \partial^2_{xx} C_E(t, S_t).$$

$$C_E(t, S_t) = S_t \cdot \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - e^{-r(T-t)} \cdot K \cdot \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

$$d_1 = \frac{\ln S_t/K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = \frac{\ln S_t/K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \cdot \sqrt{T-t}}.$$

$$\Delta = \partial_x C_E(t, S_t) = N(d_1) + S_t \cdot \partial_x N(d_1) - e^{-r(T-t)} \cdot K \cdot \partial_x N(d_2)$$

$$= N(d_1) + S_t \cdot \frac{\partial \int_{-\infty}^{d_1} f_Z(z) dz}{\partial x} - e^{-r(T-t)} \cdot K \cdot \frac{\partial \int_{-\infty}^{d_2} f_Z(z) dz}{\partial x}$$

$$= N(d_1) + S_t \cdot f_Z(d_1) \cdot \frac{\partial d_1}{\partial x} - e^{-r(T-t)} K \cdot f_Z(d_2) \cdot \frac{\partial d_2}{\partial x}$$

$$= N(d_1) + S_t \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{1}{S_t \cdot \sigma \sqrt{T-t}} - e^{-r(T-t)} K \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{1}{S_t \sigma \sqrt{T-t}}$$

$$= N(d_1) + \frac{1}{\sqrt{2\pi} \cdot \sigma S_t \sqrt{T-t}} \left(S_t e^{-\frac{d_1^2}{2}} - K e^{-r(T-t)} e^{-\frac{d_2^2}{2}} \right).$$

$$= N(d_1)$$

$$d_1^2 - d_2^2 = (d_1 + d_2)(d_1 - d_2) = \frac{2\left(\ln S_t/K + r\cdot(T-t)\right)}{\sigma\cdot\cancel{\sqrt{T-t}}} \cdot \frac{\sigma^2\cdot\cancel{(T-t)}}{\sigma\cdot\sqrt{T-t}}$$

$$= \frac{2\cdot\left(\ln S_t/K + r(T-t)\right)}{-2}$$

$$\Gamma = \partial^2_{xx} f_c(t, S_t) = \partial_x N(d_1) = \frac{\partial \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}{\partial x}$$

$$= N'(d_1) \cdot \partial_x d_1$$

$$= N'(d_1) \frac{1}{S_t \sigma \sqrt{T-t}}$$

2.We consider an Asian option with payoff

$$g(S.) = \int_0^T S_t \, dt$$

in the Black-Scholes setting. If $V_t := E^Q\left[e^{-r(T-t)} g(S.) | S_\theta, \theta \in [0,t]\right]$
Prove

$$V_t = \begin{cases} \int_0^t S_\theta \, d\theta + S_t(T-t), & \text{if } r = 0, \\ e^{-r(T-t)} \int_0^t S_\theta d\theta + \frac{1}{r}(1 - e^{-r(T-t)}) S_t, & \text{if } r > 0. \end{cases}$$

Construct a self-financing portfolio $(\Pi_t)_{t\in[0,T]}$ s.t.

$$\downarrow$$

$\Pi_t = V_t$ for $t\in[0,T]$.

$$d\Pi_t = (\Pi_t - \phi_t S_t) r dt + \phi_t dS_t$$
$$\Leftrightarrow d\tilde{\Pi}_t = \phi_t d\tilde{S}_t.$$

Solution:

$r=0$.

Let $\phi(t) = T-t$, $\Pi_0 = V_0 = E^Q[\int_0^T S_\theta d\theta | S_0] = S_0 \cdot (T)$.

(Apply Ito formula to $(T-t)\cdot S_t$)

$$d[(T-t)S_t] = -S_t dt + (T-t) dS_t.$$

$$\Pi_t = \Pi_0 + \int_0^t (T-\theta) dS_\theta = \int_0^t d[(T-\theta)S_\theta] + \int_0^t S_\theta d\theta + \Pi_0$$

$$= (T-\theta)S_\theta \Big|_{\theta=0}^{\theta=t} + \int_0^t S_\theta d\theta + \Pi_0$$

$$= (T-t)S_t - T\cdot S_0 + \int_0^t S_\theta d\theta + \Pi_0.$$

$$= (T-t)S_t + \int_0^t S_\theta d\theta.$$

At T, $\Pi_T = \int_0^T S_\theta d\theta = g(S.)$.

$$V_t = \Pi_t = (T-t)S_t + \int_0^t S_\theta d\theta.$$

when $r>0$. Let $\phi_t = \frac{1}{r}(1-e^{-r(T-t)})$ $\phi_T = 0$.

$$\Pi_0 = V_0 = E^Q[e^{-r(T-t)} g(S.) \mid S_0] = S_0 \phi_0.$$

Apply Itô formula to $(\phi_t \tilde{S}_t)$:

$$d(\phi_t \tilde{S}_t) = -\tilde{S}_t e^{-r(T-t)} dt + \phi_t d\tilde{S}_t.$$

$$\tilde{\Pi}_t = \tilde{\Pi}_0 + \int_0^t \phi_t d\tilde{S}_t$$

$$= \tilde{\Pi}_0 + \int_0^t d(\phi_\theta \tilde{S}_\theta) + \int_0^t \tilde{S}_\theta e^{-r(T-\theta)} d\theta.$$

$$= \Pi_0 + \phi_\theta \tilde{S}_\theta \Big|_{\theta=0}^{\theta=t} + \int_0^t \tilde{S}_\theta e^{-r(T-\theta)} d\theta$$

$$= \underbrace{\Pi_0}_{\phi_0 S_0} + \phi_t \tilde{S}_t - \phi_0 S_0 + \int_0^t \tilde{S}_\theta e^{-r(T-\theta)} d\theta$$

$$= \phi_t \tilde{S}_t + \int_0^t \tilde{S}_\theta e^{-r(T-\theta)} d\theta.$$

$$\Pi_T = e^{rT}(\tilde{\Pi}_T) = e^{rT}\left(\phi_T \tilde{S}_T + \int_0^T e^{-r\theta} S_\theta e^{-r(T-\theta)} d\theta\right.$$

$$= \phi_T S_T + \int_0^T S_\theta d\theta = \int_0^T S_\theta d\theta = G(S.)$$

$$\Rightarrow \Pi_t = V_t = e^{rt}\tilde{\Pi}_t = e^{rt}\cdot\left(\phi_t \tilde{S}_t + \int_0^t e^{-r\theta}\cdot S_\theta \cdot e^{-r(T-\theta)} d\theta\right)$$

$$= \phi_t S_t + e^{-r(T-t)} \int_0^t S_\theta d\theta$$

$$= \frac{1}{r}\left(1 - e^{-r(T-t)}\right) S_t + e^{-r(T-t)} \int_0^t S_\theta d\theta$$

Alternative method: $S_\theta = S_0 \cdot e^{(r-\frac{\sigma^2}{2})\theta + \sigma \cdot B_\theta^Q}$.

$$V_t = E^Q\left[e^{-r(T-t)} \int_0^T S_\theta d\theta \,\middle|\, S_\theta, \theta\in[0,t]\right]$$

$$= e^{-r(T-t)} E^Q\left[\int_0^t S_\theta d\theta + \int_t^T S_\theta d\theta \,\middle|\, S_\theta, \theta\in[0,t]\right]$$

$$= e^{-r(T-t)} \int_0^t S_\theta d\theta + e^{-r(T-t)} E^Q\left[\int_t^T S_\theta d\theta \,\middle|\, S_\theta, \theta\in[0,t]\right]$$

$$= e^{-r(T-t)} \int_0^t S_\theta d\theta + e^{-r(T-t)} E^Q\left[\int_t^T S_t \cdot e^{(r-\frac{\sigma^2}{2})(\theta-t) + \sigma(B_\theta - B_t)} d\theta \,\middle|\, S_\theta, \theta\in[0,t]\right]$$

$$= e^{-r(T-t)} \int_0^t S_\theta d\theta + e^{-r(T-t)} S_t \int_t^T E^Q\left[e^{(\quad)}\right] d\theta.$$

$$= \swarrow \quad + e^{-r(T-t)} S_t \int_t^T e^{r(\theta-t)} d\theta$$

$$= \begin{cases} e^{-r(T-t)} \int_0^t S_\theta d\theta + e^{-r(T-t)} S_t \cdot \dfrac{e^{r(T-t)} - 1}{r}, & r>0 \\ e^{-r(T-t)} \int_0^t S_\theta d\theta + S_t \cdot (T-t). & r=0. \end{cases}$$

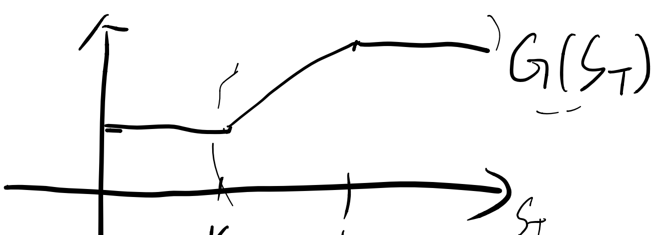
3. Let us consider the standard Black-Scholes model, where the interest rate is $r > 0$, the stock price follows the dynamic

$$dS_t = \mu S_t \, dt + \sigma S_t dB_t.$$

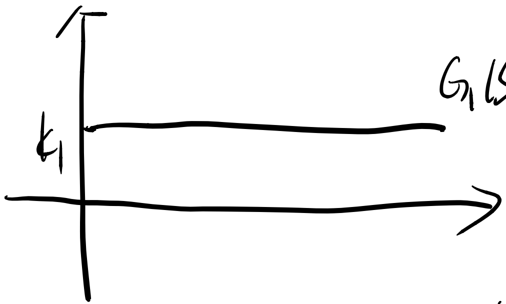
In above, B_t is the Brownian motion, μ, σ and the current price S_0 are given positive constants. Let us consider an option with the final payoff

$$G(S_T) = \begin{cases} K_1, \text{ if } 0 < S_T \le K_1, \\ S_T, \text{ if } K_1 < S_T < K_2, \\ K_2, \text{ if } K_2 \le S_T. \end{cases}$$

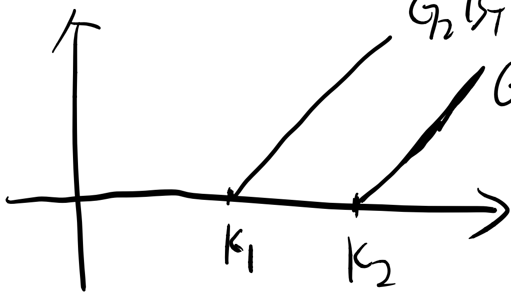
Here $0 < K_1 < K_2$ are given positive constants. Use the risk neutral evaluation method to find the current price formula for this option. (Express the result using the distribution function $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$).



Solution:

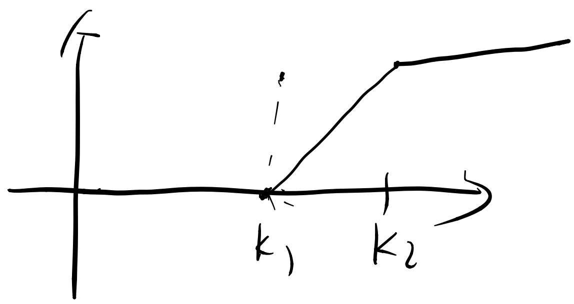


$G_1(S_T) = K_1$.



$G_2(S_T) = (S_T - K_1)_+$

$G_3(S_T) = (S_T - K_2)_+$.



$$G(S_T) = G_1(S_T) + G_2(S_T) - G_3(S_T).$$

Current price: $K_1 \cdot e^{-rT} + S_0 \cdot N(d_1^{K_1}) - K_1 e^{-rT} \cdot N(d_2^{K_1}) - (S_0 \cdot N(d_1^{K_2}) - K_2 \cdot e^{-rT} \cdot N(d_2^{K_2}))$.

$\square$.

Carr Madan formula: $g \in C^2$ . . . . .